

COMPLEMENTS OF HYPERSURFACES, VARIATION MAPS AND MINIMAL MODELS OF ARRANGEMENTS

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To Alexandru Dimca and Ștefan Papadima, on the occasion of a great anniversary

ABSTRACT. We prove the minimality of the CW-complex structure for complements of hyperplane arrangements in \mathbb{C}^n by using the theory of Lefschetz pencils and results on the variation maps within a pencil of hyperplanes. This also provides a method to compute the Betti numbers of complements of arrangements via global polar invariants.

1. INTRODUCTION

To study the topology of the complement $\mathbb{C}^n \setminus V$ of an affine hypersurface $V \subset \mathbb{C}^n$ one employs Morse theory, see for instance Randell [Ra], or the Lefschetz method of scanning by pencils of hyperplanes, as done e.g. by Dimca and Papadima in [DP]. Both methods yield in particular a CW-complex model of the complement $\mathbb{C}^n \setminus V$. It was proved in the above two papers that whenever V is a union of hyperplanes, then there exists a CW-complex model which is *minimal*, in the sense that the number of q -cells equals the Betti number $b_q(\mathbb{C}^n \setminus V)$, for any q . This notion of minimality was introduced by Papadima and Suciu in [PS] for studying the higher homotopy groups of complements of hyperplane arrangements. We give here a new proof of the minimality by using another method. We first prove the following result:

Theorem 1.1. *Let \mathcal{A} be an affine arrangement of hyperplanes, not necessarily central. Let $V_{\mathcal{A}} \subset \mathbb{C}^n$ denote the union of hyperplanes in \mathcal{A} and let \mathcal{H} be a generic hyperplane with respect to \mathcal{A} . Then, one has the isomorphisms of \mathbb{Z} -modules: $H_j(\mathbb{C}^n \setminus V_{\mathcal{A}}) \simeq H_j(\mathcal{H} \setminus V_{\mathcal{A}} \cap \mathcal{H})$, for $j \leq n - 1$ and $H_n(\mathbb{C}^n \setminus V_{\mathcal{A}}) \simeq H_n(\mathbb{C}^n \setminus V_{\mathcal{A}}, \mathcal{H} \setminus V_{\mathcal{A}} \cap \mathcal{H})$.*

Moreover, the complement $\mathbb{C}^n \setminus V_{\mathcal{A}}$ has a minimal model.

Our alternate proof uses the behaviour of the *variation maps* within a pencil of hyperplanes on $\mathbb{C}^n \setminus V_{\mathcal{A}}$. It is based on a particular case (see Theorem 3.1) of a general result on vanishing cycles of pencils, which involves variation maps, proved in [Ti3, Ti6]. We discuss in §2 some aspects of the topology of pencils on complements of affine hypersurfaces, extracted from a general theory of *non-generic Lefschetz pencils* of hypersurfaces, which we have developped in a series of papers [Ti3, Ti4, Ti6, Ti5, Ti7]. In this context, we also give a method to compute inductively the betti numbers of complements of arrangements by using global polar invariants [Ti2].

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This question was brought to our attention by Ștefan Papadima in spring 2000 in connexion with [PS] (a preprint at that time) and with the earlier paper [Ti2] in which we construct CW-complex models for affine hypersurfaces by using pencils and global polar curves (see §3.3). This note was essentially written in 2003 but not published ever since. However, we think that it might be still of current interest also because of the recent proof by J. Huh [Huh] of a conjecture about the *polar degree* stated by Dimca and Papadima [DP], of which one of the main ingredients is the non-generic Lefschetz pencil theory which we also use here.

2. COMPLEMENTS OF HYPERSURFACES

Let $V = \{f = 0\}$ be a hypersurface in \mathbb{C}^n . The complement $\mathbb{C}^n \setminus V$ is a Stein manifold, since it can be viewed as the hypersurface $\{tf(x) = 1\}$ in \mathbb{C}^{n+1} . It therefore has the homotopy type of a CW-complex of dimension $\leq n$, by Hamm's result [Ha1]. For a generic hyperplane $\mathcal{H} \in \mathbb{C}^n$ we have that the pair $(\mathbb{C}^n \setminus V, \mathcal{H} \setminus V \cap \mathcal{H})$ is $(n-1)$ -connected, by Lefschetz type results [Ha1, Ha2], see also [Ti5, Thm.4.1]. One has the following well-known consequence:

Proposition 2.1. *The space $\mathbb{C}^n \setminus V$ is obtained, up to homotopy type, by attaching to the slice $\mathcal{H} \setminus V \cap \mathcal{H}$ a certain number of n -cells.* \square

In general, hypersurface complements do not have minimal models: examples are given in [PS], one of the simplest being the case of the plane cusp $V = \{x^2 - y^3 = 0\}$.

It has been observed that the topology of the complement $\mathbb{C}^n \setminus V$ depends on the singularities of V and also on their *position*, see [Li1, Li2]. Moreover, if V is not (stratified) transversal to the hyperplane at infinity, then the non-transversality points may influence the topology, see [Li1], [LT].

2.1. A new viewpoint appeared more recently [Ti4, LT]: consider a polynomial function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ of which V is a fiber, and relate the topology of the complement to the singularities of f . It is shown in [LT] that one has two situations: either V is a general fiber of f or a special one. For some fixed f , special (or “atypical”) fibers are finitely many and have either singularities in \mathbb{C}^n or have, in some sense, singularities “at infinity” (see e.g. [ST1]). We have:

Proposition 2.2. [LT] *Let V be a general fiber of some polynomial $f : \mathbb{C}^n \rightarrow \mathbb{C}$. Then $\mathbb{C}^n \setminus V$ is homotopy equivalent to the wedge $S^1 \vee S(V)$, where $S(V)$ denotes the suspension over V . The cup-product in the cohomology ring of $\mathbb{C}^n \setminus V$ is trivial.*

Even if in the above statement V is non-singular, the complexity of the singularities at infinity of the polynomial f influences the topology of V (see [ST1], [LT] for examples). In certain situations, the general fiber of a polynomial function may be a bouquet of spheres of dimension $n-1$. It is the case when f has isolated singularities at infinity. We send to [Ti5] for a survey and more bibliography on singularities at infinity of polynomials.

When V is an atypical fiber of a polynomial, we have the following result.

Proposition 2.3. [LT] *Let $V = f^{-1}(0)$ be an atypical fiber of the polynomial function $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$. If the general fibre of f is s -connected, $s \geq 2$, then $\pi_i(\mathbb{C}^{n+1} \setminus V) = 0$, for $1 < i \leq s$, and $\pi_1(\mathbb{C}^{n+1} \setminus V) = \mathbb{Z}$. \square*

In particular, if f has “isolated singularities at infinity” then, the above discussion yields that $\pi_i(\mathbb{C}^{n+1} \setminus V) = 0$ for $1 < i \leq n - 1$.

3. VARIATION MAPS OF PENCILS OF AFFINE HYPERSURFACES

3.1. Pencils with isolated singularities. The two methods of investigating the topology of complements, by Morse functions or by Lefschetz pencils, are actually close in spirit. The latter allows one to use the full power of complex geometry and we shall stick to it in this paper.

In several recent papers we have introduced and used a general concept of *non-generic pencils of hypersurfaces* (e.g. [Ti1, Ti4, Ti5]), which may have singularities in the axis. Here we only use pencils of hyperplanes and with “no singularities in the axis”, as we describe in the following.

We consider our complement $\mathbb{C}^n \setminus V$ as embedded into the projective space \mathbb{P}^n , and we identify it to $\mathbb{P}^n \setminus (\bar{V} \cup H^\infty)$, where \bar{V} denotes the projective closure of the affine hypersurface $V = f^{-1}(0)$ and H^∞ is the hyperplane at infinity of \mathbb{P}^n . Then consider the following pencil of hyperplanes:

$$(1) \quad l(x) - tx_0 = 0,$$

where $t \in \mathbb{C}$, $l : \mathbb{C}^n \rightarrow \mathbb{C}$ is a linear function and x_0 is the coordinate at infinity of \mathbb{P}^n . This pencil defines a holomorphic function $t := l/x_0$ on $\mathbb{C}^n = \mathbb{P}^n \setminus H^\infty$, where H^∞ denotes the hyperplane at infinity. Such a pencil is not generic with respect to the divisor $\bar{V} \cup H^\infty$ since the axis $A := \{l = x_0 = 0\}$ is included into H^∞ and hence A is not transversal to any Whitney stratification of the pair $(\mathbb{P}^n, \bar{V} \cup H^\infty)$. Nevertheless, we show that this pencil is *without singularities in the axis*, in the sense of [Ti4, Definitions 2.2, 2.3].

The projective hypersurface $\bar{V} \cup H^\infty$ has a canonical minimal Whitney stratification, which we denote by \mathcal{W} . In particular, the intersection $\bar{V} \cap H^\infty$ is a union of strata. Then we consider the product stratification $\mathcal{W} \times \mathbb{C}$ in the product space $\mathbb{P}^n \times \mathbb{C}$.

By a Bertini type result, there is a Zariski-open dense set $\Omega \subset \check{\mathbb{P}}^{n-1}$ of linear forms $l : \mathbb{C}^n \rightarrow \mathbb{C}$ such that, for any $l \in \Omega$, the projective hyperplane $\{l = 0\} \subset H^\infty \simeq \mathbb{P}^{n-1}$ is transversal within H^∞ to all strata included into $\bar{V} \cap H^\infty$. In particular $\{l = 0\}$ avoids all point-strata inside H^∞ .

For such $l \in \Omega$, the hyperplane $\mathbb{H} \subset \mathbb{P}^n \times \mathbb{C}$ defined by the equation (1) is transversal within $\mathbb{P}^n \times \mathbb{C}$ to all product-strata included into $H^\infty \times \mathbb{C}$. Then the stratification $\mathcal{W} \times \mathbb{C}$ induces a stratification on \mathbb{H} , call it \mathcal{S} , which is also Whitney, by the transversality of the intersection.

Moreover, \mathcal{S} has the property that all its strata which are included into $H^\infty \times \mathbb{C}$ have a product structure, by the line \mathbb{C} . It then follows that each member of the pencil (i.e. for fixed $t \in \mathbb{C}$) is transversal to all strata of \mathcal{S} included into $H^\infty \times \mathbb{C}$. Equivalently, the projection to \mathbb{C} has no stratified singularities in the neighbourhood of $\mathbb{H} \cap (H^\infty \times \mathbb{C})$. In such a case we say that the pencil (1) has no singularities in the axis. It follows that this

pencil can have singularities only outside the axis and that they are isolated. Namely, there are finitely many points on V where the projection to the second factor $p : \mathbb{H} \rightarrow \mathbb{C}$ has a stratified singularity, with respect to the stratification \mathcal{S} . The set of these points will be denoted by $\text{Sing}_{\mathcal{S}}p$.

3.2. Variation maps. We recall from [Ti3, Ti6, Ti7] and adapt to our case the construction of the global variation maps associated to a pencil. Let us fix some notation. Let $X := \mathbb{C}^n \setminus V$ and note that X can be identified to $\mathbb{H} \cap ((\mathbb{C}^n \setminus V) \times \mathbb{C})$.

For any $M \subset \mathbb{C}$, we denote $\mathbb{H}_M := p^{-1}(M)$ and $X_M := \mathbb{H}_M \cap ((\mathbb{C}^n \setminus V) \times \mathbb{C})$. Let $\text{Sing}_{\mathcal{S}}p = \cup_{i,j} \{a_{ij}\}$, where $\Lambda := p(\text{Sing}_{\mathcal{S}}p) = \{a_1, \dots, a_p\}$ and a_{ij} denotes some point of $\text{Sing}_{\mathcal{S}}p \cap p^{-1}(a_i)$.

For $c \in \mathbb{C} \setminus \Lambda$ we say that \mathbb{H}_c , resp. X_c , is a *general fiber* of $p : \mathbb{H} \rightarrow \mathbb{C}$, resp. of $p| : \mathbb{H} \cap ((\mathbb{C}^n \setminus V) \times \mathbb{C}) \rightarrow \mathbb{C}$. Indeed, $p|$ can be identified to $l| : \mathbb{C}^n \setminus V \rightarrow \mathbb{C}$ and X_c is just $l^{-1}(c) \cap X$.

At some singularity $a_{ij} \in V$, we choose a ball B_{ij} centered at a_{ij} . For a small enough radius of B_{ij} , this is a "Milnor ball" of the holomorphic function p at a_{ij} . Next we may take a small enough disc $D_i \subset \mathbb{C}$ at $a_i \in \mathbb{C}$, so that (B_{ij}, D_i) is Milnor data for p at a_{ij} . Moreover, we may do this for all (finitely many) singularities in the fiber \mathbb{H}_{a_i} , keeping the same disc D_i , provided it is small enough.

Now the restriction of p to $\mathbb{H}_{D_i} \setminus \cup_j B_{ij}$ is a trivial fibration over D_i . One may construct a stratified vector field which trivializes this fibration and such that this vector field is tangent to the boundaries of the balls $\mathbb{H}_{D_i} \cap \partial \bar{B}_{ij}$. Using this, we may also construct a geometric monodromy of the fibration $p| : \mathbb{H}_{\partial \bar{D}_i} \rightarrow \partial \bar{D}_i$ over the circle \bar{D}_i , such that this monodromy is the identity on the complement of the balls, $\mathbb{H}_{\partial \bar{D}_i} \setminus \cup_j B_{ij}$. The same is then true, when replacing $\mathbb{H}_{\partial \bar{D}_i}$ by $\mathbb{X}_{\partial \bar{D}_i}$.

Fix some point $c_i \in \partial \bar{D}_i$. We have the geometric monodromy representation:

$$\rho_i : \pi_1(\partial \bar{D}_i, c_i) \rightarrow \text{Iso}(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}),$$

where $\text{Iso}(\cdot, \cdot)$ denotes the group of relative isotopy classes of stratified homeomorphisms (which are C^∞ along each stratum). It follows that the geometric monodromy restricted to $X_{c_i} \setminus \cup_j B_{ij}$ is the identity.

As shown above, we may identify the fiber $X_{c_i} \setminus \cup_j B_{ij}$ to the fiber $X_{a_i} \setminus \cup_j B_{ij}$ in the trivial fibration over D_i . Furthermore, in local coordinates at a_{ij} , X_{a_i} is a germ of a complex analytic space; hence, for a small enough ball B_{ij} , the set $B_{ij} \cap X_{a_i}$ retracts to $\partial \bar{B}_{ij} \cap X_{a_i}$, by the local conical structure of analytic sets [BV]. Therefore X_{a_i} is homotopy equivalent, by retraction, to $X_{a_i} \setminus \cup_j B_{ij}$.

Notation Due to the above homotopy equivalences, we shall freely use $X_{a_i}^*$ as notation for $X_{c_i} \setminus \cup_j B_{ij}$ whenever we consider the pair (X_{c_i}, X_{a_i}) .

It then follows that the geometric monodromy induces an algebraic monodromy, in any dimension q :

$$\nu_i : H_q(X_{c_i}, X_{a_i}^*; \mathbb{Z}) \rightarrow H_q(X_{c_i}, X_{a_i}^*; \mathbb{Z}),$$

such that the restriction $\nu_i : H_q(X_{a_i}^*) \rightarrow H_q(X_{a_i}^*)$ is the identity.

Consequently, any relative cycle $\delta \in H_q(X_{c_i}, X_{a_i}^*; \mathbb{Z})$ is sent by the morphism $\nu_i - \text{id}$ to an absolute cycle. In this way we define a *variation map*, for any $q \geq 0$:

$$(2) \quad \text{var}_i : H_q(X_{c_i}, X_{a_i}^*; \mathbb{Z}) \rightarrow H_q(X_{c_i}; \mathbb{Z}).$$

Variation morphisms are basic ingredients in the description of the behaviour of vanishing cycles of global and local fibrations at singular fibers of holomorphic functions, see e.g. [Mi], [La], [Si], [Ti1, 4.4]. Zariski already used $\nu_i - \text{id}$ in dimension 2, in his well-known theorem for the fundamental group. We shall use of [Ti6, Theorem 4.4] in the following form adapted to our particular case.

Theorem 3.1. [Ti3, Ti6] *Let $V \subset \mathbb{C}^n$, $l \in \Omega$ and let X_c be a general member of the pencil, as above. Then $H_q(X, X_c) = 0$ for $q \leq n - 1$ and the kernel of the surjection $H_{n-1}(X_c) \twoheadrightarrow H_{n-1}(X)$ is generated by the images of the variation maps var_i , for $i = \overline{1, p}$.*

The first claim is also a consequence of the connectivity result stated in Proposition 2.1. The second claim is highly nontrivial and is proved in [Ti6]. All the assumptions made in [Ti6, Theorem 4.4] are clearly verified, except of one, which we still need to verify: $H_q(X_c, X_{a_i}^*) = 0$ for $q \leq n - 2$. This is indeed true by the following reason. In [Ti6, 3.7, 3.9] it is shown that the named condition is satisfied whenever $H_q(X_D, X_c) = 0$ for $q \leq n - 1$, where D is a small enough disc centered at some value $a \in \Lambda$. But the later condition is fulfilled by our [Ti1, Corollary 2.7], which is based on Hamm and Lê's results in [HL].

3.3. Number of cells and polar invariants. Vanishing cycles in a pencil of hypersurfaces have been investigated in large generality, for example in [ST1, Ti1, Ti2, Ti5]. If the hypersurface $V \subset \mathbb{C}^n$ is given by $f = 0$ then, for some linear function l , one defines the global polar variety:

$$\Gamma(l, f) := \text{closure}\{\text{Sing}(l, f) \setminus \text{Sing}f\} \subset \mathbb{C}^n.$$

By the *global polar curve lemma* [Ti1, Lemma 2.4], it follows that $\Gamma(l, f)$ is either empty or it is a curve, provided that l is general enough. This means that l can be taken out of a Zariski-open set $\tilde{\Omega} \subset \Omega \subset \check{\mathbb{P}}^{n-1}$, see *loc.cit.*) Global polar curves appeared for the first time in [Ti1] in the study of the topology at infinity of polynomial functions. Local polar varieties have been introduced by Lê D.T. and B. Teissier and are currently used in the literature. We refer the reader to [Ti2, ST2, Ti5] for different aspects of global polar curves.

By [Ti1, Theorem 4.6] and especially [Ti5, Corollary 4.3] we have that the Betti number $b_n(X, X_c)$ is equal to $\lambda := \sum_{i=1}^p \lambda_{a_i}$, where λ_{a_i} is the polar number at the atypical value of the pencil a_i . According to [Ti2, Definition 3.5], λ_{a_i} is a non-negative integer equal to the following difference of intersection multiplicities:

$$\lambda_{a_i} = \text{int}(\Gamma(l, f), X_{c_i}) - \text{int}(\Gamma(l, f), X_{a_i}),$$

where c_i is a nearby typical value of the pencil.

The difference of intersection numbers appears as follows. First observe that $\Gamma(l, f)$ does not intersect some small neighbourhood of $\bar{V} \cap H^\infty$. Next, the curve $\Gamma(l, f)$ is algebraic, therefore it intersects V at a finite number of points. It is a general fact proved by Lê D.T [Le] that these points are among the stratified singularities of the restriction of the

function l to V . On the other hand, these singularities are isolated since l is general. In the particular case of arrangements of hyperplanes, the stratified singularities of the restriction $l|_V$ are precisely the 0-dimensional strata of the canonical Whitney stratification of V . So the polar curve is eventually non-empty in the neighbourhood of these points. As c_i tends to a_i , the points of intersection of $\Gamma(l, f)$ with X_{c_i} , in some neighbourhood of some 0-dimensional stratum of V which is also on \bar{X}_{a_i} , tend to this point-stratum. Consequently, there is loss of intersection multiplicity from $\text{int}(\Gamma(l, f), X_{c_i})$ to $\text{int}(\Gamma(l, f), X_{a_i})$ and this loss is localized near the point-strata of V .

Moreover, the space X is obtained from the slice X_c by attaching cells of dimension n only, by [Ti5], [ST3], see also [Ti7, Theorem 9.3.1]. We thus have a geometric interpretation of the topological quotient space X/X_c as a bouquet of $\lambda = b_n(X, X_c)$ n -spheres. By repeated slicing we get similar formulas in lower dimensions. In case of complements of hyperplane arrangements, we shall in the next section that the relative betti number $b_n(X, X_c)$ equals the absolute betti number $b_n(X)$.

4. PROOF OF THEOREM 1.1

We proceed by induction on the dimension. Our arrangement of hyperplanes \mathcal{A} defines a natural Whitney stratification $\mathcal{W}' = \{W_B\}_{B \subset \mathcal{A}}$ on $V_{\mathcal{A}}$ which is also the coarsest one. More explicitly, the strata are defined as follows. Let V_B denote the intersection of all hyperplanes corresponding to the indices of some subset $B \subset \mathcal{A}$. Then $W_B := V_B \setminus \bigcup_{C \subsetneq B} V_C$. This stratification is Whitney since along any stratum W_B , by some analytic local change of coordinates, the space $V_{\mathcal{A}}$ has the product structure $\{\text{transversal slice}\} \times W_B$.

Since the hyperplane at infinity $H^\infty \subset \mathbb{P}^n$ is transversal to all the strata, the induced natural stratification on $\bar{V}_{\mathcal{A}} \cap H^\infty$ is Whitney and it is the coarsest one. This is what we have denoted by \mathcal{W} in §3.1.

Let $l \in \Omega$ define a generic pencil of hyperplanes in \mathbb{C}^n , as in §3.1. We have seen before that the genericity of the pencil amounts to the condition that the axis of the pencil $A = \{l = 0\} \cap H^\infty$ is transversal to all the strata of \mathcal{W} .

Let \mathcal{H} denote a generic member of the pencil. By Proposition 2.1, we get that the long exact sequence of the pair $(\mathbb{C}^n \setminus V_{\mathcal{A}}, \mathcal{H} \setminus V_{\mathcal{A}} \cap \mathcal{H})$ splits into the isomorphisms $H_j(\mathbb{C}^n \setminus V_{\mathcal{A}}) \simeq H_j(\mathcal{H} \setminus V_{\mathcal{A}} \cap \mathcal{H})$, for $j \leq n - 1$, and the following exact sequence:

$$(3) \quad \begin{aligned} 0 \rightarrow H_n(\mathbb{C}^n \setminus V_{\mathcal{A}}) &\rightarrow H_n(\mathbb{C}^n \setminus V_{\mathcal{A}}, \mathcal{H} \setminus V_{\mathcal{A}} \cap \mathcal{H}) \rightarrow \\ &\rightarrow H_{n-1}(\mathcal{H} \setminus V_{\mathcal{A}} \cap \mathcal{H}) \xrightarrow{\iota_*} H_{n-1}(\mathbb{C}^n \setminus V_{\mathcal{A}}) \rightarrow 0. \end{aligned}$$

We claim that ι_* is injective. By Theorem 3.1 we have that $\ker \iota_* = \sum_{i=1}^p \text{im}(\text{var}_i)$. In our case, we may show that var_i is *trivial*, for any i . Our pencil has no singularities in the axis, it is a pencil of hyperplanes and $V_{\mathcal{A}}$ is a union of hyperplanes too. It follows that the singularities of the pencil are exactly the point-strata of the canonical stratification \mathcal{W} of $V_{\mathcal{A}}$. Then the atypical members of the pencil are those which pass through such points. The pencil can be chosen generic enough such that each member of it contains at most one such point-stratum.

Let us focus on some atypical value a_i . We may assume, without affecting the generality, that the singularity of \bar{X}_{a_i} is the origin of \mathbb{C}^n . Consider the map germ $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $x \mapsto x \exp(2i\pi t)$ for any coordinate x . Taking t as parameter, this defines a family of diffeomorphisms which preserve the arrangement $V_{\mathcal{A}}$ and its complement $\mathbb{C}^n \setminus V_{\mathcal{A}}$, and moves the hyperplane X_{c_i} of our pencil into the hyperplane $X_{\exp(2i\pi t)c_i}$, over the circle $\partial \bar{D}_i \subset \mathbb{C}$. For $t = 1$, this yields a geometric monodromy of X_{c_i} around the value a_i , at the origin of \mathbb{C}^n .

By its definition, this geometric monodromy is the identity on the hyperplane X_{c_i} and therefore also on $X_{a_i}^* \subset X_{c_i}$ (see the definition of the notation $X_{a_i}^*$ at §3.2). It then follows (from the definition of the variation map, see 3.2) that the variation of this monodromy is trivial, i.e. $\text{im}(\text{var}_i) = 0$. We have proved in this way that $\ker \iota_* = 0$, which also means that the above exact sequence (3) splits in the middle. This proves the second part of our first statement.

By the attaching result discussed at the end of §3 (see also [Ti7, Theorem 9.3.1]) we get that the number of the n -cells attached to $\mathcal{H} \setminus V_{\mathcal{A}} \cap \mathcal{H}$ in order to obtain $\mathbb{C}^n \setminus V_{\mathcal{A}}$ is equal to $b_n(\mathbb{C}^n \setminus V_{\mathcal{A}})$. The minimal model claim follows then by iterated slicing.

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